Modeling Severity Risk under PD-LGD Correlation

Chulwoo Han

*a Durham Business School, Mill Hill Lane, Durham DH1 3LB, UK
+44 191 334 5892
chulwoo.han@durham.ac.uk

Abstract

In this article, I develop a generic severity risk model in which LGD is dependent upon PD in an intuitive manner. By modeling the conditional mean of LGD as a function of PD, which also varies with systemic risk factors, this model allows an arbitrary functional relationship between PD and LGD. Based on this generic framework, several specifications of stochastic LGD are proposed with detailed calibration method. By combining these models with an extension of CreditRisk+, a versatile mixed Poisson credit risk model that is capable of handling both risk factor correlation and PD-LGD dependency is developed. An efficient simulation algorithm based on importance sampling is also introduced for risk calculation. Applied to a model portfolio, my model behaves as intended and shows the significance of severity risk. Empirical studies suggest that ignoring or incorrectly specifying severity risk can significantly underestimate credit risk, and banks need to consider this in their credit risk modeling and downturn LGD estimation. Associating the severity risk model with other credit risk frameworks such as structural model would be an interesting research topic for the future.

Keywords: Severity risk; Loss given default; Credit risk; CreditRisk+
1. Introduction

Recent studies have shown evidence that suggests loss given default is not only volatile but also positively correlated with default rates. The PD-LGD dependency can exacerbate the loss due to default in a recession period when the overall default risk is high. The use of “downturn” LGD in the Basel Capital Accord is also a reflection of this phenomenon. Frye (2000b) examines US corporate bonds and finds significant positive relationship between LGD and default rates. Madan et al. (2006) develop debt models which impose this positive relationship and obtain supporting results when applied to BBB-rated corporate bonds. Carey and Gordy (2004) find that although the relation is less obvious during low default periods, it becomes more apparent when the default rate is high, which is indeed the period of interest. Altman et al. (2005) also find positive correlation between LGD and default rates. However, they are skeptical of the existence of a macroeconomic risk factor that explains this relationship. A comprehensive review of research on PD-LGD relationship and recent developments in severity risk modeling can be found in Altman (2010), which is continuously updated reflecting latest developments and data.

Empirical evidence has led to developments of severity risk models that address the dependency between PD and LGD. Frye (2000a) models the value of collateral, hence LGD, as a linear function of a systemic risk factor, which also governs default rates. Pykhtin (2003), meanwhile, assumes that the value of collateral is an exponential function of a systemic risk factor and therefore log-normally distributed. Düllmann and Trapp (2004) build their model based on Frye (2000a) and Pykhtin (2003) but utilize logit function to ensure that LGD lies in $[0,1]$. Tasche (2004) also takes a similar approach with others but uses beta distribution for LGD. More recently, Van Damme (2011) generalizes Tasche (2004)'s work and develops a generic framework for stochastic LGD modeling.

While these models differ in specification of LGD, they are all based on Merton (1974)'s structural model framework. On the other hand, LGD modeling for another popular credit risk framework, CreditRisk+, has received far less attention and it is rare to find a model that incorporates PD-LGD dependency. One of the reasons is perhaps because people try to preserve its analytic tractability, which is often claimed to be a major benefit of CreditRisk+. It seems impossible without strong restrictions to allow LGD dependent upon PD within the CreditRisk+ framework, while maintaining its analytic tractability. Gordy (2003) incorporates severity risk by assuming that LGD is independently gamma distributed and ob-
tains loss distribution using saddlepoint approximation. He also shows that the increase of risk due to LGD volatility vanishes quickly as the number of obligors grows. Bürgisser et al. (2001) introduce stochastic variation of loss using two factors, an obligor specific factor and a systemic factor which are independent of each other. The systemic factors induce pairwise correlation between obligors but fail to incorporate correlation between PD and LGD by assuming that these factors are independent of the systemic factors that govern PD.

Despite strong evidence of PD-LGD dependency and developments of such models from academia, currently available commercial credit risk packages still rely on constant or independent distribution assumption of LGD. For example, CreditMetrics by MSCI, perhaps the most well-known commercial credit risk system based on the structural model, utilizes independent beta distribution to describe recovery rate volatility. However, as evidenced in the empirical studies of this paper and several other studies, these assumptions do not address severity risk adequately. While CreditMetrics is sold as a black box, CreditRisk+ has not been commercialized and those banks who use CreditRisk+ or its variants mostly have access to the core engine of the system and are able to modify it. Also, as demonstrated by Han (2014), CreditRisk+ has a more flexibility to accommodate various shapes of loss distribution. These facts make development of a proper severity risk model under CreditRisk+ framework particularly interesting.

The aim of the paper is two folds. Firstly, I develop a new framework for severity risk, in which the mean of LGD is assumed to vary with PD, which also varies with risk factors. The framework allows nonlinear relationship between PD and LGD such as power function that shows the best fit in Altman et al. (2005). Based on the framework, several specifications of stochastic LGD are considered and evaluated through empirical studies. Secondly, I combine these models with the common factor CreditRisk+ by Han and Kang (2008) to develop a new credit risk model that is capable of incorporating both risk factor correlation and severity risk within CreditRisk+ framework. The common factor CreditRisk+ is particularly well suited for this purpose as it assumes a macroeconomic risk factor that affects all assets. For this, analytic tractability is abandoned and instead a simulation method based on importance sampling is introduced. Equipped with high performance computers, there is no reason to adhere to an analytic solution sacrificing flexibility. As demonstrated later in this article, the simulation method turns out to be not only accurate but also very efficient. To my best knowledge, this paper is the first of its kind in both ways, i.e., allowing nonlinear PD-LGD relationship in severity risk modeling and
incorporating dependent severity risk with the CreditRisk+ framework.

The rest of this article is organized as follows. In Section 2, the common factor CreditRisk+ that incorporates risk factor correlation is briefly introduced. Section 3 is devoted to development of stochastic LGD models. Detailed calibration method of each model is proposed at the end of the section. Applied to a model portfolio, the models are evaluated in several ways in Section 4. The efficiency of the simulation algorithm is also assessed in this section. Concluding remarks and suggestions are given in Section 5 and the simulation algorithm for loss distribution is illustrated in Appendix A.

2. The Common Factor CreditRisk+

In the original CreditRisk+ model of CSFP (1997), PD is assumed to be a linear function of the risk factors, i.e.,

\[ P_i = PD_i \left( w_{0i} + \sum_{k=1}^{K} w_{ki} X_k \right) \]  

(1)

where \( X = \{X_1, \ldots, X_K\} \) are gamma distributed independent risk factors with mean 1 and variance \( \sigma_X^2 \), and the sum of the weights, \( \sum_{k=0}^{K} w_{ki} \), is equal to 1. Even though it is theoretically possible to incorporate asset correlation in CreditRisk+ by appropriately choosing the weights, defining independent risk factors and assigning weights on them is difficult and impractical. For this reason, extensions of the original model that explicitly take the correlation into account have been introduced, and one of the latest developments is the common factor CreditRisk+ model (CreditRisk++) by Han and Kang (2008). CreditRisk++ assumes that a correlated risk factor can be decomposed into a sector specific factor \( Y_k \) and a macroeconomic factor \( \hat{Y} \) that are independent of each other, i.e.,

\[ X_k = \delta_k Y_k + \gamma_k \hat{Y}, \quad k = 1, \ldots, K \]  

(2)

where

\[ Y_k \sim \text{Gamma}(\theta_k, 1) \]  

(3)

\[ \hat{Y} \sim \text{Gamma}(\hat{\theta}, 1) \]  

(4)

Then, the probability of default can be rewritten as a linear combination of \( K + 1 \) gamma distributed independent risk factors, \( \hat{X}_k \).

\[ P_i = PD_i \left( w_{0i} + \sum_{k=1}^{K+1} w_{ki} \hat{X}_k \right) \]  

(5)
where

\[ \hat{X}_k \sim \text{Gamma}(\theta_k, \delta_k), \quad k = 1, \ldots, K, \quad (6) \]

\[ \hat{X}_{K+1} \sim \text{Gamma}\left(\bar{\theta}, 1\right), \quad \text{and} \]

\[ w_{K+1,i} = \sum_{k=1}^{K} w_{ki} \gamma_k. \quad (8) \]

\( \hat{X}_k, \ k = 1, \ldots, K \) are sector specific risk factors and \( \hat{X}_{K+1} \) is a macroeconomic risk factor that has influence on all sectors. The degree of influence is determined by \( \delta_k \) and \( \gamma_k \). The expected values and covariance matrix of the correlated risk factors, \( \hat{X}_k \), have the form

\[ E(X_k) = \delta_k \theta_k + \gamma_k \bar{\theta}, \quad (9) \]

\[ V(X_k) = \delta_k^2 \theta_k + \gamma_k^2 \bar{\theta}; \quad (10) \]

\[ \text{COV}(X_k, X_l) = \gamma_k \gamma_l \bar{\theta}. \quad (11) \]

Appropriately choosing the parameter values, various covariance structures can be described by CreditRisk++. The model can be calibrated by minimizing the distance between observed covariance matrix and the above covariance matrix equation subject to \( E(X_k) = 1 \). In general, matching the variance terms exactly and then minimizing the distance between covariance terms yields a more stable result. The main advantage of CreditRisk++ is that it can incorporate risk factor correlations in a very flexible and intuitive manner, while maintaining the framework of CreditRisk+. Therefore, most of the numerical algorithms and extensions developed for CreditRisk+ can be reused without modification.

3. Modeling Severity Risk

In the CreditRisk+, LGD is assumed constant. However, as mentioned earlier, there is strong evidence that LGD is not only stochastic, but more importantly, correlated with PD. To take this into consideration, LGD is assumed to have a beta distribution whose mean is dependent upon risk factors. More specifically, the following specification for loss given default \( U_i \) is considered.

\[ U_i|X = \text{Beta}(a_i, b_i) \quad (12) \]
with
\[
E(U_i|X) = LGD_i \left( \phi_{0i} + \frac{1 - \phi_{0i}}{1 - w_{0i}} \sum_{k=1}^{K} w_{ki}X_k \right)
\]
\[
= LGD_i \left( \phi_{0i} + \sum_{k=1}^{K} \phi_{ki}X_k \right) \tag{13}
\]

where 0 ≤ φ_{0i} ≤ 1 and \(\phi_{ki} = (1 - \phi_{0i})/(1 - w_{0i})w_{ki}\).\(^1\) That is, the conditional mean of LGD is assumed to be determined by the systemic risk factors with the same relative weights as the conditional PD. Put another way, the conditional mean of LGD is assumed to be a linear function of conditional PD as Equation (13) can be rewritten as
\[
E(U_i|X) = \frac{\phi_{0i} - w_{0i}}{1 - w_{0i}} LGD_i + \frac{1 - \phi_{0i}}{1 - w_{0i}} \frac{LGD_i}{PD_i} \cdot P_i. \tag{14}
\]

This specification ensures that the unconditional mean of LGD satisfies
\[
E(U_i) = E[E(U_i|X)] = LGD_i, \tag{15}
\]

and \(U_i\) and \(P_i\) are positively correlated. One problem with this specification is the conditional mean can exceed 1, in which case the following adjustment should be applied.
\[
E(U_i|X) = \min(E(U_i|X), 1) \tag{16}
\]

This can be prevented in most situations by setting \(\phi_{0i}\) sufficiently large, which is plausible since the mean itself would not have a large variance. Empirical studies in Section 4 confirm this. Another specification that is worth consideration is to utilize logit function in order to ensure \(E(U_i|X) \in [0, 1]\):
\[
E(U_i|X) = \frac{1}{1 + \exp \left( - \phi_{0i} - \sum_{k=1}^{K} \phi_{ki}X_k \right)} \tag{17}
\]

However, this specification does not have analytic mean and variance of \(U_i\), and a numerical integration should be employed for calibration. For this reason, I discard this specification and adopt an alternative specification as illustrated below.

\(^1\)Since CreditRisk++ can be represented within the framework of CreditRisk+, \(K\) independent risk factors, \(X_k \sim \text{Gamma}(\alpha_k, \beta_k)\) are assumed for generality.
3.1. Alternative Specifications

The LGD model described above nests two special cases. When $\phi_{0i} = 1$, LGD becomes an independent, beta distributed stochastic variable which is the same assumption as the one in CreditMetrics. On the other hand, when $a_i + b_i \to \infty$, i.e., $V(U_i|X) \to 0$, the model becomes equivalent to modeling $U_i$ as a linear function of the risk factors

$$U_i = LGD_i \left( \phi_{0i} + \sum_{k=1}^{K} \phi_{ki} X_k \right). \quad (18)$$

This representation requires only one variable, $\phi_{0i}$, to be estimated. Also, since LGD can theoretically exceed 1 when costs are included, the bound condition of $[0, 1]$ can be loosened.

Altman et al. (2005) find that PD-LGD relation is best fitted by a power function. Inspired by this, and also to substitute the logit specification in (17), following specifications are also considered.

$$E(U_i|X) = \phi_{0i} P_i^{\phi_{1i}} \quad (19)$$

and

$$E(U_i|X) = \frac{1}{1 + \exp(-\phi_{0i} - \phi_{1i} P_i)} \quad (20)$$

where

$$P_i = PD_i \left( w_{0i} + \sum_{k=1}^{K} w_{ki} X_k \right)$$

That is, the conditional mean of LGD is directly modeled as a function of the conditional PD. In this way, any functional form other than those above can also be implemented.

3.2. Model Calibration

For the specification in (13), there are three parameters, $\phi_0$, $a$, and $b$, to estimate.\(^2\) Assuming that the following values are available, a two-step approach is proposed:

$$LGD = E(U), \quad VLGD = V(U), \quad CPDLGD = COV(P, U).$$

\(^2\)For notational convenience, the subscript $i$ is omitted here unless clarification is required. It is also true that the parameters are likely to be estimated at a portfolio level rather than for individual debts.
3.2.1. Step 1: Estimation of $\phi_0$

$\phi_0$ is estimated from the covariance between PD and LGD, $CPDLGD$.

$$CPDLGD = E(P \cdot U) - E(P)E(U) = E(P \cdot U) - PD \cdot LGD$$  \hspace{1cm} (21)$$

From (1) and (13), we have

$$E(P \cdot U) = E\left[PD \left(w_0 + \sum_{k=1}^{K} w_k X_k\right) \cdot LGD\left(\phi_0 + \sum_{k=1}^{K} \phi_k X_k\right)\right]$$

$$= PD \cdot LGD \cdot E\left[w_0 \phi_0 + \sum_{k=1}^{K} (w_0 \phi_k + \phi_0 w_k) X_k + \sum_{k=1}^{K} \sum_{l=1}^{K} w_k \phi_l X_k X_l\right]$$

$$= PD \cdot LGD \cdot \left[w_0 \phi_0 + \sum_{k=1}^{K} (w_0 \phi_k + \phi_0 w_k) \mu_{X_k} + \sum_{k=1}^{K} w_k \phi_k (\sigma^2_{X_k} + \mu^2_{X_k})\right]$$  \hspace{1cm} (22)$$

where $\mu_{X_k}$ and $\sigma^2_{X_k}$ are mean and variance of $X_k$. When $X_k \sim \text{Gamma}(\alpha_k, \beta_k)$,

$$\mu_{X_k} = \alpha_k \beta_k, \quad \sigma^2_{X_k} = \alpha_k \beta_k^2$$

By combining (21) and (22) and from the relation $\phi_k = (1 - \phi_0)/(1 - w_0) w_k$, we obtain

$$\phi_0 = \frac{CPDLGD - PD \cdot LGD \cdot E\left[w_0 \phi_0 + \sum_{k=1}^{K} (w_0 \phi_k + \phi_0 w_k) \mu_{X_k} + \sum_{k=1}^{K} w_k \phi_k (\sigma^2_{X_k} + \mu^2_{X_k})\right]}{w_0 - \frac{1}{1 - w_0} \sum_{k=1}^{K} w_k \theta_k \beta_k (-1 + 2w_0 + w_k \delta_k (1 + \alpha_k))}$$  \hspace{1cm} (23)$$

3.2.2. Step 2: Estimation of $(a, b)$

Once $\phi_0$ is obtained, the parameters of the beta distribution can be estimated from the mean and variance of LGD. As the conditional mean is determined by the systemic risk factors, it is convenient to parametrize the beta distribution using its mean $\mu_\beta$ and sample size $\nu_\beta$:

$$\mu_\beta = \frac{a}{a + b}, \quad \nu_\beta = a + b.$$  \hspace{1cm} (24)$$

Using $\mu_\beta$ and $\nu_\beta$, the variance has the form

$$\sigma^2_\beta = \frac{ab}{(a + b)^2(a + b + 1)} = \frac{\mu_\beta (1 - \mu_\beta)}{1 + \nu_\beta}$$  \hspace{1cm} (25)$$
Therefore, the conditional mean and variance of LGD can be written as

\[ E(U|X) = \mu_\beta = LGD \left( \phi_0 + \sum_{k=1}^{K} \phi_k X_k \right) \]  

(26)

\[ V(U|X) = \sigma_\beta^2 = \frac{\mu_\beta(1 - \mu_\beta)}{1 + \nu_\beta} \]  

(27)

By the law of total variance, the unconditional variance of LGD can be decomposed into two parts.

\[ VLGD = E[V(U|X)] + V[E(U|X)] \]

\[ = E(\mu_\beta(1 - \mu_\beta)) + V(\mu_\beta) \]

\[ = \frac{E(\mu_\beta) - E(\mu_\beta)^2}{1 + \nu_\beta} + \frac{\nu_\beta}{1 + \nu_\beta} V(\mu_\beta) \]  

(28)

\[ = \frac{LGD - LGD^2}{1 + \nu_\beta} + \frac{\nu_\beta}{1 + \nu_\beta} LGD^2 \sum_{k=1}^{K} \phi_k^2 \sigma_{X_k}^2 \]

Therefore, we obtain

\[ \nu_\beta = -\frac{VLGD - LGD + LGD^2}{VLGD - LGD^2 \sum_{k=1}^{K} \phi_k^2 \sigma_{X_k}^2} \]  

(29)

Then, \( a \) and \( b \) are obtained from

\[ a = \mu_\beta \nu_\beta, \quad b = (1 - \mu_\beta) \nu_\beta. \]  

(30)

3.2.3. Calibration of the Alternative Models

For the limiting case in (18), only \( \phi_0 \) needs to be estimated. In this case, I suggest to calibrate the model using \( V(U) \) rather than \( COV(PD, LGD) \) as the former is likely to be estimated with higher accuracy and more easily available.

\[ VLGD = LGD^2 \sum_{k=1}^{K} \phi_k^2 \sigma_{X_k}^2 \]  

(31)

Since \( \phi_k = \frac{(1-\phi_0)}{(1-w_0)} w_k \), \( \phi_0 \) can be obtained from the equation

\[ (1 - \phi_0)^2 = \frac{VLGD}{LGD^2 \sum_{k=1}^{K} \left( \frac{w_k}{1-w_0} \right)^2 \sigma_{X_k}^2} \]  

(32)
For the nonlinear specifications in (19) and (20), $\phi_0$ and $\phi_1$ can be obtained directly from a regression of LGD on PD using their time series data and applying the associated functional form. Then, only $\nu_\beta$ of beta distribution remains to be estimated. One might use $VLGD$ to estimate $\nu_\beta$ as in (28). But this method is rather complicated as $V(\mu_\beta)$ is not analytically tractable. Alternatively, a constant value might be assigned based on some judgment. In the empirical studies in the next section, I use $\nu_\beta$ obtained from (29) for the linear model.

4. Empirical Studies

4.1. The Portfolio

A test portfolio is constructed based on the actual market data; bond characteristics are set by referring to Moody’s (2012) annual report. PD and its standard deviation of each rating are assumed based on the annual issuer-weighted corporate default rates reported in Exhibit 30 of the report. LGD of each rating is based on the average senior unsecured bond recovery rates in Exhibit 21. As recovery rate time series for each rating are not available, LGD standard deviation of all bonds calculated from the annual defaulted corporate bond and loan recoveries in Exhibit 20 is identically applied to all ratings. These values are summarized in Table 1.

<table>
<thead>
<tr>
<th>Rating</th>
<th>PD</th>
<th>SPD</th>
<th>LGD</th>
<th>SLGD</th>
<th>EaD/ Num.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aa</td>
<td>0.06</td>
<td>0.19</td>
<td>63.00</td>
<td>9.50</td>
<td>100.00</td>
</tr>
<tr>
<td>A</td>
<td>0.10</td>
<td>0.27</td>
<td>68.00</td>
<td>9.50</td>
<td>100.00</td>
</tr>
<tr>
<td>Baa</td>
<td>0.27</td>
<td>0.47</td>
<td>59.00</td>
<td>9.50</td>
<td>100.00</td>
</tr>
<tr>
<td>Ba</td>
<td>1.07</td>
<td>1.65</td>
<td>53.00</td>
<td>9.50</td>
<td>100.00</td>
</tr>
<tr>
<td>B</td>
<td>3.42</td>
<td>4.03</td>
<td>62.00</td>
<td>9.50</td>
<td>100.00</td>
</tr>
<tr>
<td>Caa-C</td>
<td>13.77</td>
<td>17.03</td>
<td>64.00</td>
<td>9.50</td>
<td>100.00</td>
</tr>
</tbody>
</table>

Table 1: The model portfolio. The portfolio consists of 1,000 bonds distributed evenly across 10 industry groups in Table 2 and distributed across ratings according to the composition shown in the last column. Each bond has a notional value of 100 million dollars. PD, SPD, LGD, and SLGD are respectively probability of default, its standard deviation, loss given default, and its standard deviation, all in percentage values, and they are based on Exhibit 20, 21, and 30 of Moody’s (2012).

Based on the debts outstanding in the market, the portfolio allocation among ratings is assumed as shown in the last column of Table 1. Since
there is no default record of Aaa corporates since 1920, Aaa rated bonds are excluded from the investment set. The notional value of corporate bonds appears to vary widely from a few million to a few billion dollars. For simplicity, EaD of all bonds is set to 100 million dollars. Also, each bond is assumed to be issued by different issuers, i.e., default of a bond occurs independently from each other.

Among Moody’s 11 broad industry groups, the portfolio consists of bonds from 10 industries except ‘Government Related Issuers’ since default in this group is extremely rare and reliable statistics cannot be obtained. More specifically, it is assumed that the rating portfolio of 100 bonds described in Table 1 is invested in each of 10 broad industry groups resulting in the whole portfolio consisting of 1,000 bonds distributed evenly among the industry groups and distributed among ratings within each industry according to the composition in Table 1. The 10 industry groups and their descriptive statistics are reported in Table 2.

<table>
<thead>
<tr>
<th>Corr.</th>
<th>I1</th>
<th>I2</th>
<th>I3</th>
<th>I4</th>
<th>I5</th>
<th>I6</th>
<th>I7</th>
<th>I8</th>
<th>I9</th>
<th>I10</th>
</tr>
</thead>
<tbody>
<tr>
<td>I1</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I2</td>
<td>0.47</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I3</td>
<td>0.62</td>
<td>0.84</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I4</td>
<td>-0.04</td>
<td>0.26</td>
<td>0.13</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I5</td>
<td>0.45</td>
<td>0.65</td>
<td>0.59</td>
<td>-0.01</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I6</td>
<td>0.51</td>
<td>0.76</td>
<td>0.61</td>
<td>0.12</td>
<td>0.60</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I7</td>
<td>0.35</td>
<td>0.55</td>
<td>0.57</td>
<td>0.02</td>
<td>0.19</td>
<td>0.31</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I8</td>
<td>0.19</td>
<td>0.67</td>
<td>0.50</td>
<td>0.28</td>
<td>0.32</td>
<td>0.59</td>
<td>0.51</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I9</td>
<td>0.45</td>
<td>0.59</td>
<td>0.52</td>
<td>0.28</td>
<td>0.24</td>
<td>0.43</td>
<td>0.58</td>
<td>0.50</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>I10</td>
<td>0.44</td>
<td>0.36</td>
<td>0.34</td>
<td>0.17</td>
<td>0.01</td>
<td>0.29</td>
<td>0.53</td>
<td>0.42</td>
<td>0.51</td>
<td>1.00</td>
</tr>
<tr>
<td>Mean</td>
<td>0.41</td>
<td>1.69</td>
<td>1.97</td>
<td>1.35</td>
<td>0.86</td>
<td>2.67</td>
<td>2.30</td>
<td>1.32</td>
<td>2.49</td>
<td>0.17</td>
</tr>
<tr>
<td>Std</td>
<td>0.72</td>
<td>2.03</td>
<td>2.09</td>
<td>1.70</td>
<td>2.62</td>
<td>3.86</td>
<td>2.30</td>
<td>1.90</td>
<td>2.98</td>
<td>0.27</td>
</tr>
</tbody>
</table>


4.2. PD-LGD Relation

Mean and standard deviation of PD and LGD of all bonds, and their correlation are computed using the time series data of 1982-2011 in Exhibit 20.
and 30 of the Moody’s report, and are reported in Table 3. The correlation between PD and LGD is very high with correlation coefficient 0.71 during the period. This is in line with the high correlation of 0.75 during 1982-2001 reported by Altman et al. (2005). Regression results of LGD against PD using different functions are also reported in the same table and displayed in Figure ?? and 1. Three functions, linear, power, and logit functions are considered. The power function fits the data best with R-squared value of 0.56, but all three functions show similar fitting abilities with R-squared larger than 0.5. Altman et al. (2005) also find that power function slightly outperforms other functions in terms of R-squared and Altman (2010) shows that out-of-samples are also placed near the regression line.

<table>
<thead>
<tr>
<th></th>
<th>PD</th>
<th>LGD</th>
<th>Regression</th>
<th>$\phi_0$</th>
<th>$\phi_1$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.67</td>
<td>58.46</td>
<td>Linear</td>
<td>0.487</td>
<td>5.851</td>
<td>0.539</td>
</tr>
<tr>
<td>Std</td>
<td>1.20</td>
<td>9.53</td>
<td>Power</td>
<td>1.291</td>
<td>0.187</td>
<td>0.564</td>
</tr>
<tr>
<td>Corr</td>
<td>0.71</td>
<td></td>
<td>Logit</td>
<td>-0.067</td>
<td>25.434</td>
<td>0.550</td>
</tr>
</tbody>
</table>

Table 3: Descriptive statistics of PD and LGD of all bonds and regression results. The statistics are calculated from default rates and recovery rates of all bonds during 1982-2011, excerpted from Exhibit 20 and 30 of Moody’s (2012).

4.3. Simulation Efficiency

Credit risk of the portfolio is computed using CreditRisk++ with different severity risk models. 11—10 industry and 1 macroeconomic—independent systemic risk factors are extracted from the 10 correlated industry factors and loss distribution is generated from a Monte-Carlo simulation utilizing importance sampling as described in Appendix A.

To address accuracy of the simulation method, it is first compared with an analytic method for the case of constant LGD. For analytic calculation, the numerical method proposed by Haaf et al. (2004) is employed, and the simulation results are obtained from 10,000 iterations. Simulation is repeated 100 times to compute the variance of simulation error. In important sampling, the predetermined portfolio loss is set to 6,000, which is approximately the loss at 99.9% quantile. The results are reported in Table 4. It is remarkable how significantly simulation error is reduced via importance sampling. While the usual Monte-Carlo simulation has simulation error over 10% at higher quantiles, e.g., RMSE of 14.93% at 99.99%, the error is generally less than 1% when importance sampling is utilized. It is also
Figure 1: Regression of LGD on PD. In (a), y is LGD and x is PD, and in (b), y is \( \log(\text{LGD}/(1-\text{LGD})) \) and x is PD.
notable that importance sampling reduces the error not only around the predetermined loss level, but over a wide range of quantile including the expected loss. Computation cost-wise, simulation does not take more time compared to the analytic method at least for the portfolio under consideration. Indeed, the naive simulation takes a shorter time (0.86 seconds) than the analytic method (1.02 seconds). Of course, the computation cost of simulation methods will linearly increase with the size of the portfolio, while that of analytic method will increase more slowly. Nevertheless, considering the simulation is programmed with Matlab and run in a desktop environment, computational burden should by no means be a barrier for adopting a simulation-based algorithm even for a large size portfolio.

<table>
<thead>
<tr>
<th></th>
<th>Analytic Mean</th>
<th>Simulation RMSE</th>
<th>Simulation MAE</th>
<th>Analytic Mean</th>
<th>Simulation RMSE</th>
<th>Simulation MAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>EL</td>
<td>790.64</td>
<td>791.33</td>
<td>0.87</td>
<td>0.70</td>
<td>790.31</td>
<td>0.72</td>
</tr>
<tr>
<td>UL99</td>
<td>2948.83</td>
<td>2946.42</td>
<td>3.97</td>
<td>3.10</td>
<td>2952.60</td>
<td>1.34</td>
</tr>
<tr>
<td>UL99.9</td>
<td>5593.14</td>
<td>5524.52</td>
<td>6.19</td>
<td>4.77</td>
<td>5587.07</td>
<td>1.08</td>
</tr>
<tr>
<td>UL99.99</td>
<td>8404.32</td>
<td>7904.53</td>
<td>14.93</td>
<td>12.56</td>
<td>8405.61</td>
<td>0.74</td>
</tr>
<tr>
<td>Elapsed</td>
<td>1.02</td>
<td>0.86</td>
<td>1.04</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Simulation error of risk measures. Analytic, Simulation, and Simulation (IS) respectively represent analytic method, naive Monte-Carlo simulation and Monte-Carlo simulation with importance sampling. The predetermined loss for important sampling is set at 6,000. Simulation errors are computed from 100 simulation runs of 10,000 iterations. RMSE: Root mean squared error (%), MAE: Mean absolute error (%), and Elapsed: elapsed time (seconds).

To assess convergence of simulation error, simulation is also run for different numbers of iterations. The results are displayed in Figure 2, where RMSE of UL99.9 is plotted against the number of iterations. The figure reveals the effect of importance sampling more clearly. Simulation error of importance sampling is surprisingly small even for a small number of iterations. For example, simulation with 5,000 iterations for a portfolio of 1,000 bonds has RMSE of only 1.16%. This suggests that when the size of the portfolio grows, the number of iterations does not need to be increased at the same speed.

4.4. Model Calibration and Diagnostics

Contribution of severity risk on the total credit risk is assessed using the five LGD models that are described in Section 3, i.e., independently distributed beta (Beta Indep), beta distribution with mean as a linear function
Figure 2: Convergence of simulation error. x-axis represents number of iterations and y-axis RMSE(%) of UL99.9. No IS: Naive Monte-Carlo simulation, and IS: Monte-Carlo simulation with importance sampling.

of conditional PD (Beta Linear), beta distribution with mean as a power function of condition PD (Beta Power), beta distribution with mean as a logit function of conditional PD (Beta Logit), and finally linear function of conditional PD, the limiting case of Beta Linear (Linear). These models are compared with constant LGD model (Const).

For Beta Indep model, the parameters of the beta distribution are computed for each rating from the mean and standard deviation of LGD in Table 1. For Beta Linear model, the 2-step approach is applied. When parameters are estimated for each bond, i.e., each industry and rating, parameters of some bonds turn out to be counterintuitive or out of the bounds, e.g., negative $\nu_\beta$. One of the reasons is because while the covariance between PD and LGD is assumed constant across all bonds, PD and LGD differ among bonds. Therefore, instead of PD and LGD of individual bonds, those of all bonds reported in Table 3 are used in Equation (23). Since LGD differs across ratings while $\sigma_{X_k}^2$ differs across industries, $\nu_\beta$ from Equation (29) sometime has a negative value. To prevent this, $\phi_k$ is averaged across all bonds before put in Equation (29). For Beta Power and Beta Logit models, the regression parameters in Table 3 are used as the estimates of $\phi_0$ and $\phi_1$, and $\nu_\beta$ of Beta Linear model is reused. Since the regressions are run on the overall PD and LGD time series and the results represent not cross-sectional variation but time series variation, using individual PD in (19) and (20) is not appropriate. Thus, the following manipulation is applied to
generate conditional mean of LGD. First, conditional PD is generated using the overall PD, $\hat{PD}$

$$P_i = \hat{PD} \left( w_{0i} + \sum_{k=1}^{K} w_{ki} X_k \right)$$

Then, conditional mean of LGD is calculated, for example, in case of power function, from

$$E(U_i|X) = \frac{LGD_i}{\hat{LGD}} \cdot \phi_0 P_i^{\phi_1}$$

The mean PD and LGD of all bonds in Table 3 are used for $\hat{PD}$ and $\hat{LGD}$, respectively. Finally, $\phi_0$ in Linear model is estimated from the variance of LGD of each bond using Equation (32).

Calibration results are reported in Table 5. In the table, Min, Max, and Mean are computed across bonds. $\nu_\beta$ of Beta Indep is smaller than that of Beta Linear, because the total variance of LGD is solely explained by beta distribution when LGD is assumed independent. As a result, beta distribution derived from independence assumption has a wider distribution as illustrated in Figure 3. Another notable thing in the results is $\phi_0$ of Beta Linear and Linear. Even though it is estimated from different informations, i.e., PD-LGD covariance for Beta Linear and LGD variance for Linear, the estimates are very similar. The models are diagnosed by comparing simulated LGD values to the true LGD of the portfolio. MLGD is average LGD from a simulation with 10,000 iterations and SLGD is standard deviation. For example, Mean of MLGD is average of average LGD of each bond. The simulated LGDs are very close to the true LGD in all models except Beta Power. In fact, the parameters of Beta Power and Beta Logit are not estimated in a way that the mean LGD is matched to the true LGD, and there is no guarantee that the average of the simulated LGD should be similar to the true LGD. Still, it is remarkable that the simulated LGD of Beta Logit is very close to the true LGD. The difference is more apparent in Beta Power due to the asymmetry of power function. The standard deviations of LGD are also reasonably close to the true value.

4.5. Effects of Severity Risk

Table 6 reports risk calculation of each LGD model. The results are also illustrated in Figure 4. Change is percentage change from the constant LGD case, and RMSE is the root mean square error of simulation. Importance sampling works remarkably well even when stochastic LGD is introduced.
<table>
<thead>
<tr>
<th></th>
<th>Beta</th>
<th>Beta</th>
<th>Beta</th>
<th>Beta</th>
<th>Linear</th>
<th>Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Indep</td>
<td>Linear</td>
<td>Power</td>
<td>Logit</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>Min</td>
<td>0.686</td>
<td>1.291</td>
<td>0.067</td>
<td>0.800</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Max</td>
<td>0.952</td>
<td>1.291</td>
<td>0.067</td>
<td>0.919</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.861</td>
<td>1.291</td>
<td>0.067</td>
<td>0.874</td>
<td></td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>Min</td>
<td>0.187</td>
<td>25.434</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Max</td>
<td>0.187</td>
<td>25.434</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>0.187</td>
<td>25.434</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu_\beta$</td>
<td>Min</td>
<td>23.111</td>
<td>43.227</td>
<td>43.227</td>
<td>43.227</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>24.758</td>
<td>54.081</td>
<td>54.081</td>
<td>54.081</td>
<td></td>
</tr>
<tr>
<td>MLDG</td>
<td>Min</td>
<td>52.786</td>
<td>52.300</td>
<td>43.040</td>
<td>51.518</td>
<td>53.000</td>
</tr>
<tr>
<td></td>
<td>Max</td>
<td>68.296</td>
<td>68.109</td>
<td>65.912</td>
<td>67.883</td>
<td>67.986</td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>62.300</td>
<td>62.022</td>
<td>56.169</td>
<td>61.449</td>
<td>62.095</td>
</tr>
</tbody>
</table>

Table 5: Calibration results of the severity risk models. MLDG and SLGD are average and standard deviation of LGD of each bond computed from a simulation with 10,000 iterations, except for those in the last column which are true portfolio values. Min, Max, and Mean are computed across the bonds in the portfolio.
Figure 3: Beta distribution for the estimates of $\nu_\beta$ at different values of mean, $\mu_\beta$. 

(a) Beta Indep ($\nu_\beta = 24.76$) 

(b) Beta Linear ($\nu_\beta = 54.08$) 

Figure 3: Beta distribution for the estimates of $\nu_\beta$ at different values of mean, $\mu_\beta$. 

18
From the results of Beta Indep, you can see that idiosyncratic severity risk is mostly diversified away and the risk measures are almost identical to those of the constant LGD. This has been reported in other studies; for example, Gordy (2003) shows that idiosyncratic severity risk vanishes very quickly and it becomes trivial even for a small size portfolio with a few hundreds of bonds. When the PD-LGD dependency is incorporated, however, the contribution of severity risk becomes significant: 99.9% UL is increased by about 60% in all models, and the increase becomes more significant at a higher quantile. This suggests that assuming LGD as a constant or a stochastic but independent variable can significantly underestimate risk. The results could be used as a guidance to set downturn LGD; a normal LGD might be adjusted proportionately to the increase of risk from the constant case. All LGD models that incorporate PD-LGD dependency surprisingly result in similar risk measures. Beta Logit and Linear models yield higher risk measures below 99.9% level but all four models have similar values at higher levels. Also note that not only unexpected loss but also expected loss is, though at a lesser degree, significantly increased: at least 16.0% (Beta Linear) up to 21.3% (Beta Logit).

<table>
<thead>
<tr>
<th></th>
<th>Const</th>
<th>Beta Indep</th>
<th>Beta Linear</th>
<th>Beta Power</th>
<th>Beta Logit</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>EL</td>
<td>Mean</td>
<td>791</td>
<td>792</td>
<td>917</td>
<td>929</td>
<td>959</td>
</tr>
<tr>
<td></td>
<td>Change</td>
<td>0.12</td>
<td>15.96</td>
<td>17.55</td>
<td>21.30</td>
<td>17.46</td>
</tr>
<tr>
<td></td>
<td>RMSE</td>
<td>0.82</td>
<td>0.79</td>
<td>0.81</td>
<td>0.95</td>
<td>0.90</td>
</tr>
<tr>
<td>UL99</td>
<td>Mean</td>
<td>2949</td>
<td>2955</td>
<td>4341</td>
<td>4275</td>
<td>4624</td>
</tr>
<tr>
<td></td>
<td>Change</td>
<td>0.21</td>
<td>47.22</td>
<td>44.96</td>
<td>56.80</td>
<td>52.14</td>
</tr>
<tr>
<td></td>
<td>RMSE</td>
<td>1.30</td>
<td>1.64</td>
<td>1.32</td>
<td>1.40</td>
<td>1.55</td>
</tr>
<tr>
<td>UL99.9</td>
<td>Mean</td>
<td>5593</td>
<td>5596</td>
<td>8866</td>
<td>8787</td>
<td>9087</td>
</tr>
<tr>
<td></td>
<td>Change</td>
<td>0.06</td>
<td>58.51</td>
<td>57.11</td>
<td>62.47</td>
<td>64.59</td>
</tr>
<tr>
<td></td>
<td>RMSE</td>
<td>0.96</td>
<td>0.88</td>
<td>1.05</td>
<td>0.94</td>
<td>0.95</td>
</tr>
<tr>
<td>UL99.99</td>
<td>Mean</td>
<td>8404</td>
<td>8407</td>
<td>13623</td>
<td>13506</td>
<td>13636</td>
</tr>
<tr>
<td></td>
<td>Change</td>
<td>0.03</td>
<td>62.10</td>
<td>60.70</td>
<td>62.25</td>
<td>63.70</td>
</tr>
<tr>
<td></td>
<td>RMSE</td>
<td>0.86</td>
<td>0.77</td>
<td>0.83</td>
<td>0.75</td>
<td>0.79</td>
</tr>
</tbody>
</table>

Table 6: Risk measures by different severity risk models. Change is change of risk measure from the constant LGD case (Const). Change and RMSE are in percentage and risk measures are in million dollars.
5. Concluding Remarks

In this article, I develop a generic severity risk model in which LGD is dependent upon PD in an intuitive manner. By modeling the conditional mean of LGD as a function of PD, which also varies with systemic risk factors, this model allows an arbitrary functional relationship between PD and LGD including nonlinear forms such as power function that is found to offer the best fit for PD-LGD covariation. Based on this generic framework, several specifications of stochastic LGD are proposed with detailed calibration method that uses easily obtainable data. By combining these models with a generalized CreditRisk+ model, a versatile mixed Poisson credit risk model is developed. This model is capable of handling various forms of PD-LGD dependency as well as risk factor correlation. This added capability comes at a cost of analytic tractability and a simulation method based on importance sampling is introduced for risk calculation. The simulation method turns out to be very accurate and efficient without any notable disadvantage.

The severity risk models are applied to a model portfolio and evaluated. The model portfolio is artificially constructed based on the actual market data. The empirical studies suggest that ignoring or incorrectly specifying, e.g., by assuming independence, severity risk can significantly underestimate credit risk: In my study, risk increases up to 60% from the case of constant LGD. Banks should recognize this and adopt a proper severity risk model to assess their credit risk more accurately or at least consider the effect of severity risk when determining downturn LGD. All models yield similar risk
measures and no particular model offers a significantly better performance. In fact, the framework behind the models is very flexible and can be easily generalized to accommodate other types of PD-LGD relationship. It could also be applied to other credit risk models such as structural models based on Merton (1974). Evaluating severity risk models thoroughly using a larger set of actual portfolios or developing credit risk models incorporating severity risk under different frameworks, all may well serve as future research topics.

References


Appendix A. Importance Sampling for Loss Distribution

When severity risk is incorporated, CreditRisk+ is no longer analytically tractable and loss distribution can only be obtained via a simulation method. Glasserman and Li (2003) develop an importance sampling technique for a mixed Poisson credit risk model and Han and Kang (2008) show that the method indeed produces a very accurate result at a low computational cost. Extension of Glasserman and Li (2003)’s work for CreditRisk+ with severity risk is straightforward and is illustrated below. As demonstrated in Section 4.3, this technique is very efficient and generates sufficiently accurate results for all severity risk models.

Suppose there are $N$ assets in the portfolio and $K$ systemic risk factors $X_k \sim \text{Gamma}(\alpha_k, \beta_k)$ which are independent of each other. Define $\theta$ as
the exponential twisting parameter for importance sampling. The cumulant generating function of the portfolio loss, \( L = \sum_{i=1}^{N} L_i \) under the framework of CreditRisk+ is given by

\[
\psi(\theta) = \psi^{(1)}(\theta) + \psi^{(2)}(\theta)
\]  

(A.1)

where

\[
\psi^{(1)}(\theta) = \sum_{i=1}^{N} P_i w_{0i} \left( e^{V_i \theta} - 1 \right), 
\]

(A.2)

\[
\psi^{(2)}(\theta) = - \sum_{k=1}^{K} \alpha_k \log \left( 1 - \beta_k \sum_{i=1}^{N} P_i w_{ki} \left( e^{V_i \theta} - 1 \right) \right)
\]

(A.3)

where \( P_i \) is PD conditional on the risk factors and \( V_i = LGD_i \cdot EaD_i \) is expected amount of loss given default. The first order derivative of \( \psi(\theta) \) with respect to \( \theta \) is

\[
\psi'(\theta) = \psi^{(1)'}(\theta) + \psi^{(2)'}(\theta)
\]

(A.4)

where

\[
\psi^{(1)'}(\theta) = \sum_{i=1}^{N} P_i w_{0i} V_i e^{V_i \theta}, 
\]

(A.5)

\[
\psi^{(2)'}(\theta) = \sum_{k=1}^{K} \frac{\alpha_k \beta_k \sum_{i=1}^{N} P_i w_{ki} V_i e^{V_i \theta}}{1 - \beta_k \sum_{i=1}^{N} P_i w_{ki} \left( e^{V_i \theta} - 1 \right)}.
\]

(A.6)

Portfolio loss simulation is performed by following the procedure.

1. Solve

\[
\psi'(\theta) = L_p
\]

(A.7)

\[
\theta = \max(0, \theta)
\]

(A.8)

for \( \theta \). \( L_p \) is a predetermined portfolio loss, e.g., Value-at-Risk, around which samples are to be drawn. For CreditRisk+, \( L_p \) can be determined from an analytic solution or by running a simulation without importance sampling.
2. Compute $\tau_k$, $k = 1, \ldots, K$ from

$$\tau_k = \sum_{i=1}^{N} P_i w_{ki} \left(e^{V_i \theta} - 1\right)$$

3. Draw samples of risk factors

$$X_k \sim \text{Gamma} \left(\alpha_k, \beta_k \frac{1}{1 - \beta_k \tau_k}\right), \ k = 1, \ldots, K.$$  

4. Compute the conditional default probabilities

$$P_i = PD_i \left(w_i + \sum_{k=1}^{K} w_{ki} X_k\right), \ i = 1, \ldots, N$$

5. Draw samples of default events

$$D_i \sim \text{Poisson} \left(P_i e^{V_i \theta}\right), \ i = 1, \ldots, N.$$  

6. Compute the conditional mean of LGD using either of the equations.

$$E(U_i | X) = LGD_i \left(\phi_{0i} + \sum_{k=1}^{K} \phi_{ki} X_k\right) \quad \text{(Beta Linear)}$$

$$E(U_i | X) = \phi_{0i} P_i^{\phi_{1i}} \quad \text{(Beta Power)}$$

$$E(U_i | X) = \frac{1}{1 + \exp(-\phi_{0i} - \phi_{1i} P_i)} \quad \text{(Beta Logit)}$$

for $i = 1, \ldots, N$. With $\mu_{\beta,i} = E(U_i | X)$, compute beta distribution parameters from

$$a_i = \mu_{\beta,i} \nu_{\beta,i}, \quad b_i = (1 - \mu_{\beta,i}) \nu_{\beta,i}.$$  

Skip this step for Beta Indep model.

7. Draw samples of loss given default.

$$U_i \sim \text{Beta}(a_i, b_i), \ i = 1, \ldots, N$$  

Skip this step for Linear model and set $U_i = E(U_i | X)$ with the first equation of step 6.
8. Portfolio loss is given by

\[ L = \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} D_i \cdot U_i \cdot EaD_i \]

And the probability associated with this loss is given by the likelihood ratio,

\[ LR = \exp(-\theta L' + \psi(\theta)) \]

where \( L' = \sum_{i=1}^{N} D_i \cdot LGD_i \cdot EaD_i \) is portfolio loss under constant LGD assumption.

9. Repeat from step 3 until the desired number of iterations is reached.